

Certain logarithmic integrals, including solution of Monthly problem #tbd, zeta values, and expressions for the Stieltjes constants

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Abstract

We solve problem x proposed by O. Oloa, AMM xxx 2012 **119?** (to appear), p. yyy for certain definite logarithmic integrals. A number of generating functions are developed with certain coefficients p_n , and some extensions are presented. The explicit relation of p_n to Nörlund numbers $B_n^{(n)}$ is discussed. Certain inequalities are conjectured for the $\{p_n\}$ sequence of coefficients, including its convexity, and an upper bound is demonstrated. It is shown that p_n values may be used to express the Stieltjes constants for the Hurwitz and Riemann zeta functions, as well as values of these zeta functions at integer argument. Other summations with the p_n coefficients are presented.

Key words and phrases

logarithmic integrals, Pochhammer symbol, generating function, digamma function, Glaisher constant, Nörlund number, Hurwitz zeta function, Stieltjes constants

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Solution of problem xxx

O. Oloa has proposed the following problem in the Amer. Math. Monthly **119** (?), yyy (2012?).

(a) Prove the formula

$$\int_0^1 \left(\frac{1}{\ln x} + \frac{1}{1-x} \right)^2 dx = \ln(2\pi) - \frac{3}{2}. \quad (1.1)$$

(b) If $\sigma \geq 0$, find a closed form expression for

$$\int_0^1 \left(\frac{1}{\ln x} + \frac{1}{1-x} \right)^2 x^\sigma dx.$$

N.B. Although the statement (b) is restricted to $\sigma \geq 0$, this part may be extended to $\operatorname{Re} \sigma > -1$, and our explicit expressions reflect this fact.

We first prove (1.1) and answer part (b), and then present several extensions.

Proof. Let $(a)_n = \Gamma(a+n)/\Gamma(a)$ be the Pochhammer symbol, where Γ is the Gamma function, $\psi = \Gamma'/\Gamma$ be the digamma function, $\gamma = -\psi(1)$ be the Euler constant, and ${}_pF_q$ the generalized hypergeometric function (e.g., [7, 1]). We introduce the positive constants (e.g., [3], Proposition 11, [4], Proposition 5, [5], Proposition 2)

$$p_{n+1} = -\frac{1}{n!} \int_0^1 (-x)_n dx = \frac{(-1)^{n+1}}{n!} \sum_{k=1}^n \frac{s(n, k)}{k+1}, \quad (1.2)$$

where $s(k, \ell)$ is the Stirling number of the first kind. The first few values of these are $p_2 = 1/2$, $p_3 = 1/12$, $p_4 = 1/24$, $p_5 = 19/720$, and $p_6 = 3/160$. These constants enter the generating function

$$\sum_{n=1}^{\infty} p_{n+1} z^{n-1} = \frac{1}{z} + \frac{1}{\ln(1-z)}, \quad |z| < 1. \quad (1.3)$$

Multiplying (1.3) by $\ln(1-z)$ and manipulating series, one finds the recursion relation

$$p_{n+1} = \frac{1}{n+1} - \sum_{j=1}^{n-1} \frac{p_{j+1}}{(n-j+1)}, \quad n \geq 1.$$

We then have

Lemma 1.

$$\frac{1}{\ln^2(1-z)} = \sum_{n=1}^{\infty} [(n+1)p_{n+3} - np_{n+2}]z^n + p_3 + \frac{(1-z)}{z^2}, \quad |z| < 1. \quad (1.4)$$

This follows from the derivative expression

$$\sum_{n=2}^{\infty} (n-1)p_{n+1}z^{n-2} = -\frac{1}{z^2} + \frac{1}{(1-z)\ln^2(1-z)}. \quad (1.5)$$

As a consequence, we obtain

$$\left(\frac{1}{\ln(1-z)} + \frac{1}{z}\right)^2 = \frac{1}{4} + \sum_{n=1}^{\infty} [(n+3)p_{n+3} - np_{n+2}]z^n. \quad (1.6)$$

Then for (a),

$$\begin{aligned} \int_0^1 \left(\frac{1}{\ln(1-z)} + \frac{1}{z}\right)^2 dz &= \int_0^1 \left(\frac{1}{\ln z} + \frac{1}{1-z}\right)^2 dz \\ &= \int_0^1 \left\{ \frac{1}{4} + \sum_{n=1}^{\infty} [(n+3)p_{n+3} - np_{n+2}](1-z)^n \right\} dz \\ &= \frac{1}{4} + \sum_{n=1}^{\infty} [(n+3)p_{n+3} - np_{n+2}] \frac{1}{n+1}. \end{aligned} \quad (1.7)$$

The integral representation of (1.2) may now be inserted, and the sums rewritten in terms of binomial coefficients. For instance, we have

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \frac{(n+3)}{(n+2)!} (-x)_{n+2} = \sum_{n=1}^{\infty} (-1)^n \frac{(n+3)}{(n+1)} \binom{x}{n+2} = \sum_{n=1}^{\infty} (-1)^n \left[1 + \frac{2}{(n+1)}\right] \binom{x}{n+2}$$

$$= \frac{1}{2}(2-x)(x-1) - x[1-2\gamma+x-2\psi(x+1)]. \quad (1.8)$$

Noting that $\int_0^1 \psi(x+1)dx = 0$, we have

$$\begin{aligned} \int_0^1 \left(\frac{1}{\ln(1-z)} + \frac{1}{z} \right)^2 dz &= \frac{1}{4} + \int_0^1 \left[\gamma - \frac{x}{2} - 2\gamma x + \frac{3}{2}x^2 - 2x\psi(x+1) \right] dx \\ &= \ln(2\pi) - \frac{3}{2}. \end{aligned} \quad (1.9)$$

In the last step, we integrated by parts (e.g., [2]),

$$\int_0^1 x\psi(x+1)dx = - \int_0^1 \ln \Gamma(x+1)dx = - \int_0^1 [\ln x + \ln \Gamma(x)]dx = 1 - \frac{1}{2} \ln(2\pi). \quad (1.10)$$

For (b), we let $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ be the Beta function. Then

$$\begin{aligned} &\int_0^1 \left(\frac{1}{\ln x} + \frac{1}{1-x} \right)^2 x^\sigma dx \\ &= \int_0^1 \left\{ \frac{1}{4} + \sum_{n=1}^{\infty} [(n+3)p_{n+3} - np_{n+2}](1-x)^n \right\} x^\sigma dx \\ &= \frac{1}{4(\sigma+1)} + \sum_{n=1}^{\infty} [(n+3)p_{n+3} - np_{n+2}]B(n+1, \sigma+1). \end{aligned} \quad (1.11)$$

Lemma 2. For $\text{Re } y > 0$,

$$\sum_{n=1}^{\infty} p_{n+3}B(n+1, y) = -\frac{1}{2} \left[\frac{1}{6y} - 1 + 2y - \ln(2\pi) + 2 \ln \Gamma(y) + (1-2y)\psi(y) \right]. \quad (1.12)$$

Proof. From (1.2) we have

$$\sum_{n=1}^{\infty} p_{n+3}B(n+1, y) = -\frac{1}{2y} \int_0^1 x(x-1) [{}_3F_2(1, 1, 2-x; 3, y+1; 1) - 1] dx. \quad (1.13)$$

We now use the identity

$$\frac{(1)_j}{(3)_j} = 2 \frac{(1)_j}{(2)_j} - \frac{(2)_j}{(3)_j}, \quad (1.14)$$

being a special case of

$$\frac{(a)_j}{(a+2)_j} = (a+1) \frac{(a)_j}{(a+1)_j} - a \frac{(a+1)_j}{(a+2)_j},$$

to obtain

$$\begin{aligned} {}_3F_2(1, 1, 2-x; 3, y+1; 1) &= \sum_{j=0}^{\infty} \frac{(1)_j}{(3)_j} \frac{(2-x)_j}{(y+1)_j} \\ &= \sum_{j=0}^{\infty} \left[2 \frac{(1)_j}{(2)_j} - \frac{(2)_j}{(3)_j} \right] \frac{(2-x)_j}{(y+1)_j} \frac{(1)_j}{j!} \\ &= \frac{2y}{x(x-1)} [1 - x - (x+y-1)\psi(y) + (x+y-1)\psi(x+y+1)]. \end{aligned} \quad (1.15)$$

Carrying out the integration of (1.13) gives the Lemma. \square

Then by Proposition 2 in the next section with $y = \sigma + 1$ we obtain

$$\int_0^1 \left(\frac{1}{\ln x} + \frac{1}{1-x} \right)^2 x^\sigma dx = (\sigma+1) \ln(\sigma+1) - 2\sigma + \sigma\psi(\sigma+1) - 2 \ln \Gamma(\sigma+1) + \ln(2\pi) - \frac{3}{2}. \quad (1.16)$$

Remarks. As is apparent from (1.11) and (1.16), $\text{Re } \sigma = -1$ is the ‘critical line’ for divergence of the integral.

We have the following

Corollary 1. For $n \geq 1$,

$$(n+3)p_{n+3} - np_{n+2} = \sum_{k=1}^{n+1} p_{k+1} p_{n-k+3}. \quad (1.17)$$

This follows from multiplication of series, using (1.6),

$$\left(\frac{1}{\ln(1-z)} + \frac{1}{z} \right)^2 = \frac{1}{4} + \sum_{n=1}^{\infty} [(n+3)p_{n+3} - np_{n+2}] z^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=1}^{n+1} p_{k+1} p_{n-k+3} z^n. \quad (1.18)$$

Is there a combinatorial interpretation of identity (1.13)?

The Appendix generalizes (1.14) and the following identity for ratios of Pochhammer symbols.

Extensions

We may proceed similarly as above, and find for instance

$$\begin{aligned} \frac{1}{\ln^3(1-z)} &= \frac{1}{2} \sum_{n=1}^{\infty} [(n+1)(n+2)p_{n+4} - (n+1)(2n+1)p_{n+3} + n^2 p_{n+2}] z^n \\ &\quad - \frac{1}{z^3} + \frac{3}{2z^2} - \frac{1}{2z}, \quad |z| < 1, \end{aligned} \quad (2.1)$$

giving, along with (1.3) and (1.4),

$$\begin{aligned} \left(\frac{1}{\ln(1-z)} + \frac{1}{z} \right)^3 &= \frac{1}{\ln^3(1-z)} + \frac{3}{z \ln^2(1-z)} + \frac{3}{z^2 \ln(1-z)} + \frac{1}{z^3} \\ &= \frac{1}{2} \sum_{n=4}^{\infty} [(n+4)(n+5)p_{n+4} - (n+1)(2n+7)p_{n+3} + n^2 p_{n+2}] z^n \\ &\quad + \frac{1}{8} + \frac{z}{16} - \frac{z^2}{24} + \frac{133}{4320} z^3. \end{aligned} \quad (2.2)$$

Then

$$\begin{aligned} \int_0^1 \left(\frac{1}{\ln(1-z)} + \frac{1}{z} \right)^3 dz &= \int_0^1 \left(\frac{1}{\ln z} + \frac{1}{1-z} \right)^3 dz \\ &= \frac{1}{2} \sum_{n=4}^{\infty} [(n+4)(n+5)p_{n+4} - (n+1)(2n+7)p_{n+3} + n^2 p_{n+2}] \frac{1}{(n+1)} + \frac{3073}{17280} \\ &= -\frac{31}{24} + 6 \ln A, \end{aligned} \quad (2.3)$$

wherein A is Glaisher's constant, such that $\ln A = -[\zeta(-1) + \zeta'(-1)] = 1/12 - \zeta'(-1)$, and $\zeta(s)$ is the Riemann zeta function. The latter contribution enters from the integral

$$\int_0^1 t^2 \psi(t+1) dt = -2 \int_0^1 t \ln \Gamma(t+1) dt = \frac{1}{2}(1 - \ln 2\pi) + 2 \ln A. \quad (2.4)$$

This integral may be readily determined from Kummer's Fourier series for $\ln \Gamma$. Otherwise, it may be found through the infinite series

$$\int_0^1 t^2 \psi(t+1) dt = -\frac{\gamma}{3} + \sum_{k=2}^{\infty} (-1)^k \zeta(k) \int_0^1 t^{k+1} dt = -\frac{\gamma}{3} + \sum_{k=2}^{\infty} \frac{(-1)^k}{k+2} \zeta(k). \quad (2.5)$$

For reference, we have from (2.1)

$$\begin{aligned} \frac{1}{\ln^4(1-z)} &= \frac{1}{6} \sum_{n=1}^{\infty} [(n+1)(n+2)(n+3)p_{n+5} - 3(n+1)^2(n+2)p_{n+4} \\ &+ (n+1)(3n^2 + 3n + 1)p_{n+3} - n^3 p_{n+2}] z^n - \frac{1}{720} - \frac{1}{6z} + \frac{7}{6z^2} - \frac{2}{z^3} + \frac{1}{z^4}, \quad |z| < 1. \end{aligned} \quad (2.6)$$

Then also using (1.3), (1.4), and (2.1), we find

$$\begin{aligned} &\left(\frac{1}{\ln(1-z)} + \frac{1}{z} \right)^4 \\ &= \frac{1}{6} \sum_{n=1}^{\infty} [(n+1)(n+2)(n+3)p_{n+5} - 3(n+1)^2(n+2)p_{n+4} + (n+1)(3n^2 + 3n + 1)p_{n+3} - n^3 p_{n+2}] z^n \\ &\quad - \frac{1}{720} + 2 \sum_{n=1}^{\infty} [(n+1)(n+2)p_{n+4} - (n+1)(2n+1)p_{n+3} + n^2 p_{n+2}] z^{n-1} \\ &\quad + 6 \sum_{n=1}^{\infty} [(n+2)p_{n+4} - (n+1)p_{n+3}] z^{n-1} + 4 \sum_{n=1}^{\infty} p_{n+4} z^{n-1}. \end{aligned} \quad (2.7)$$

By using (1.2) we obtain

$$\int_0^1 \left(\frac{1}{\ln(1-z)} + \frac{1}{z} \right)^4 dz = \int_0^1 \left(-\frac{1}{720} + \frac{53}{18}t - \frac{57}{8}t^2 + \frac{143}{36}t^3 + \frac{t^4}{24} \right)$$

$$\begin{aligned}
& + \left[\frac{1}{6} - \frac{8}{3}t + 6t^2 - \frac{10}{3}t^3 \right] [\psi(t+1) + \gamma] dt \\
& = -\frac{49}{72} + 2 \ln A + \frac{5}{2\pi^2} \zeta(3).
\end{aligned} \tag{2.8}$$

From (2.6) we have

$$\begin{aligned}
\frac{1}{\ln^5(1-z)} &= \frac{1}{24} \sum_{n=1}^{\infty} [(n+1)(n+2)(n+3)(n+4)p_{n+6} - 2(n+1)(n+2)(n+3)(2n+3)p_{n+5} \\
&+ (n+1)(n+2)(6n^2 + 12n + 7)p_{n+4} - (n+1)(2n+1)(2n^2 + 2n + 1)p_{n+3} + n^4 p_{n+2}] z^n \\
&- \frac{1}{24z} + \frac{5}{8z^2} - \frac{25}{12z^3} + \frac{5}{2z^4} - \frac{1}{z^5}, \quad |z| < 1,
\end{aligned} \tag{2.9}$$

leading to

$$\begin{aligned}
& \left(\frac{1}{\ln(1-z)} + \frac{1}{z} \right)^5 \\
&= \frac{1}{24} \sum_{n=1}^{\infty} [(n+1)(n+2)(n+3)(n+4)p_{n+6} - 2(n+1)(n+2)(n+3)(2n+3)p_{n+5} \\
&+ (n+1)(n+2)(6n^2 + 12n + 7)p_{n+4} - (n+1)(2n+1)(2n^2 + 2n + 1)p_{n+3} + n^4 p_{n+2}] z^n \\
&+ \frac{5}{6} \sum_{n=1}^{\infty} [(n+1)(n+2)(n+3)p_{n+5} - 3(n+1)^2(n+2)p_{n+4} + (n+1)(3n^2 + 3n + 1)p_{n+3} - n^3 p_{n+2}] z^{n-1} \\
&+ 5 \sum_{n=1}^{\infty} [(n+2)(n+3)p_{n+5} - (n+2)(2n+3)p_{n+4} + (n+1)^2 p_{n+3}] z^{n-1} \\
&+ 10 \sum_{n=1}^{\infty} [(n+3)p_{n+5} - (n+2)p_{n+4}] z^{n-1} + 5 \sum_{n=1}^{\infty} p_{n+5} z^{n-1}.
\end{aligned} \tag{2.10}$$

By performing the integration and using (1.2) we determine

$$\int_0^1 \left(\frac{1}{\ln(1-z)} + \frac{1}{z} \right)^5 dz = -\frac{4367}{8640} + \frac{5}{3} \ln A + \frac{15}{8\pi^2} \zeta(3) - \frac{35}{3} \zeta'(-3). \tag{2.11}$$

Here we have used

$$\int_0^1 t^4 \psi(t) dt = \frac{49}{180} - 4\zeta'(-1) + \zeta'(0) + 6\zeta'(-2) - 4\zeta'(-3)$$

$$= -\frac{11}{180} + 4 \ln A - \frac{1}{2} \ln(2\pi) - \frac{3}{2\pi^2} \zeta(3) - 4\zeta'(-3). \quad (2.12)$$

Based upon the evaluation of $\int_0^1 t^k \psi(t) dt = -k \int_0^1 t^{k-1} \ln \Gamma(t) dt$, $k > -1$, we may anticipate that $\int_0^1 \left(\frac{1}{\ln(1-z)} + \frac{1}{z} \right)^{k+1} dz$ evaluates in terms of

$$\begin{aligned} & \mathbb{Q} + \mathbb{Q} \ln A + \mathbb{Q} \frac{\zeta(3)}{\pi^2} + \mathbb{Q} \frac{\zeta(5)}{\pi^4} + \dots + \mathbb{Q} \frac{\zeta(k-1)}{\pi^{k-2}} \\ & + \mathbb{Q} \zeta'(-3) + \mathbb{Q} \zeta'(-5) + \dots + \zeta'(1-k), \quad k \text{ even}, \end{aligned} \quad (2.13a)$$

and

$$\begin{aligned} & \mathbb{Q} + \mathbb{Q} \ln A + \mathbb{Q} \frac{\zeta(3)}{\pi^2} + \mathbb{Q} \frac{\zeta(5)}{\pi^4} + \dots + \mathbb{Q} \frac{\zeta(k)}{\pi^{k-1}} \\ & + \mathbb{Q} \zeta'(-3) + \mathbb{Q} \zeta'(-5) + \dots + \zeta'(2-k), \quad k \text{ odd}. \end{aligned} \quad (2.13b)$$

We provide some discussion of intermediate calculations in the above steps. First, we consider the integrals $\int_0^1 t^k \psi(t) dt$. These may be treated by multiple integrations by parts using the fact that $-\psi(a)$ is the zeroth Stieltjes constant for the Hurwitz zeta function $\zeta(s, a)$. That is, we have the limit representation

$$\psi(t) = -\lim_{z \rightarrow 1} \left(\zeta(z, t) - \frac{1}{z-1} \right). \quad (2.14)$$

Thus,

$$\int_0^1 t^k \psi(t) dt = -\lim_{z \rightarrow 1} \int_0^1 t^k \left[\zeta(z, t) - \frac{1}{z-1} \right] dt, \quad (2.15)$$

with the interchange justified by uniform convergence of the integral. We have the properties $\partial_a \zeta(s, a) = -s \zeta(s+1, a)$ and $\int_0^1 \zeta(s, t) dt = 0$ for $\text{Re } s < 1$. By iteration we may then obtain the integrals

$$\int_0^1 t^k \zeta(z, t) dt = -\frac{1}{z-1} \int_0^1 t^k \partial_t \zeta(z-1, t) dt$$

$$= \frac{k}{z-1} \int_0^1 t^{k-1} \zeta(z-1, t) dt - \frac{\zeta(z-1)}{z-1}. \quad (2.16)$$

Secondly, the insertion of the integral representation of (1.2) for p_{n+1} into sums such as (2.10) gives certain hypergeometric summations. Again, ${}_pF_q$ denotes the generalized hypergeometric function. As an illustration, consider a contribution from a sum such as $\sum_{n=1}^{\infty} n^3 p_{n+6}$. By recalling the property $(a)_{n+1} = a(a+1)_n$, we have

$$\begin{aligned} - \sum_{n=1}^{\infty} n^3 \frac{(-t)_{n+5}}{(n+5)!} &= - \sum_{n=0}^{\infty} (n+1)^3 \frac{(-t)_{n+6}}{(n+6)!} \\ &= -t(t-1)(t-2)(t-3)(t-4)(t-5) \sum_{n=0}^{\infty} \frac{(2)_n^3 (6-t)_n}{(1)_n^3 (n+6)!} \\ &= -t(t-1)(t-2)(t-3)(t-4)(t-5) \frac{1}{720} \sum_{n=0}^{\infty} \frac{(2)_n^3 (6-t)_n}{(1)_n^2 (7)_n} \frac{1}{n!} \\ &= -t(t-1)(t-2)(t-3)(t-4)(t-5) \frac{1}{720} {}_4F_3(2, 2, 2, 6-t; 1, 1, 7; 1). \end{aligned} \quad (2.17)$$

Thirdly, (i) there are relations between the ${}_{p+1}F_p$ sums so obtained, and (ii) they are often related to the digamma function. By manipulating divided difference forms of the ψ function, relations such as the following may be obtained:

$$\begin{aligned} &{}_3F_2(2, 2, 3-t; 3, 4; 1) + {}_4F_3(2, 2, 2, 3-t; 1, 3, 4; 1) \\ &= -\frac{12[3-3t+\gamma t+t\psi(t+1)]}{t(t-1)(t-2)} + \frac{12[1-2t+\gamma t+t\psi(t+1)]}{t(t-1)(t-2)} = \frac{12}{t(t-1)}. \end{aligned} \quad (2.18)$$

Likewise, we have

$${}_5F_4(2, 2, 2, 2, 3-t; 1, 1, 3, 4; 1) + {}_4F_3(2, 2, 2, 3-t; 1, 3, 4; 1) = \frac{12(4-t)}{t(t-1)(t-2)}. \quad (2.19)$$

In this way, summations over the p_n constants may be evaluated.

We also record the following.

Lemma 3. For $\operatorname{Re} x > 0$,

$$\psi(x) - \ln x = - \sum_{n=1}^{\infty} p_{n+1} \frac{(n-1)!}{(x)_n}. \quad (2.20)$$

Corollary 2. For $\operatorname{Re} x > -1$,

$$x(\psi(x) - \ln x) = - \sum_{n=0}^{\infty} p_{n+2} \frac{n!}{(x+1)_n}. \quad (2.21)$$

Proof. For $\operatorname{Re} x > 0$,

$$\psi(x) - \ln x = \int_0^{\infty} e^{-xt} \left(\frac{1}{t} - \frac{1}{1-e^{-t}} \right) dt. \quad (2.22)$$

Let $t = -\ln(1-z)$, giving

$$\psi(x) - \ln x = - \int_0^1 (1-z)^{x-1} \left(\frac{1}{\ln(1-z)} + \frac{1}{z} \right) dz. \quad (2.23)$$

Now use the generating function (1.3), so that

$$\begin{aligned} \psi(x) - \ln x &= - \sum_{n=1}^{\infty} p_{n+1} \int_0^1 (1-z)^{x-1} z^{n-1} dz \\ &= - \sum_{n=1}^{\infty} p_{n+1} B(x, n). \end{aligned} \quad (2.24)$$

For Corollary 2, we use $x/(x)_n = 1/(x+1)_{n-1}$. \square

The proof we have given of (2.20) complements that of Proposition 5(a) of [4]. Of course this relation may be directly verified with the aid of (1.3):

$$- \sum_{n=1}^{\infty} p_{n+1} \frac{(n-1)!}{(x)_n} = \int_0^1 \sum_{n=1}^{\infty} \frac{(-t)_n}{n(x)_n} dt$$

$$= \int_0^1 [\psi(x) - \psi(t+x)] dt = \psi(x) - \ln[\Gamma(x+1)/\Gamma(x)] = \psi(x) - \ln x. \quad (2.25)$$

We may note that the representation (2.23) may be repeatedly integrated by parts.

We have for example

$$\begin{aligned} \psi(x) - \ln x &= -\frac{1}{x} \int_0^1 (1-z)^{x-1} \left[\frac{1}{\ln^2(1-z)} - \frac{1}{z^2} + \frac{1}{z} \right] dz - \frac{1}{2x} \\ &= -\frac{1}{x^2} \int_0^1 (1-z)^{x-1} \left[\frac{2}{\ln^3(1-z)} + \frac{2}{z^3} - \frac{3}{z^2} + \frac{1}{z} \right] dz - \frac{1}{2x} - \frac{1}{12x^2} \\ &= -\frac{1}{x^3} \int_0^1 (1-z)^{x-1} \left[\frac{6}{\ln^4(1-z)} - \frac{6}{z^4} + \frac{12}{z^3} - \frac{7}{z^2} + \frac{1}{z} \right] dz - \frac{1}{2x} - \frac{1}{12x^2} \\ &= -\frac{1}{x^4} \int_0^1 (1-z)^{x-1} \left[\frac{24}{\ln^5(1-z)} + \frac{24}{z^5} - \frac{60}{z^4} + \frac{57}{z^3} - \frac{22}{z^2} + \frac{1}{z} \right] dz - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4}. \end{aligned} \quad (2.26)$$

In writing these equations, we have used the values of p_2 , p_3 , p_4 , and p_5 for the boundary terms. We note that the latter terms give the asymptotic expansion of $\psi(x) - \ln x$ as $x \rightarrow \infty$. We expand on this point next with regard to Corollary 3.

The asymptotic expansion

$$\psi(z) - \ln z = -\frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}} = -\frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \dots, \quad (2.27)$$

where B_k are the Bernoulli numbers, is well known. In fact, we may readily derive it in the following manner. We have

$$\tilde{\psi}(x) \equiv \psi(x) - \ln x + \frac{1}{2x} = -\int_0^{\infty} e^{-2tx} \left(\coth t - \frac{1}{t} \right) dt. \quad (2.28)$$

The asymptotic form of the integral for large x is obtained as $t \rightarrow 0$, in which case we may use the expansion

$$\coth t - \frac{1}{t} = \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} t^{2k-1}, \quad t^2 < \pi^2. \quad (2.29)$$

Then (2.27) follows. Now we may apply Stirling's formula for Γ to Lemma 2, so that for $x \rightarrow \infty$,

$$\begin{aligned} \psi(x) - \ln x = & - \sum_{n=1}^{\infty} p_{n+1} \frac{(n-1)!}{x^n} \left[1 + \frac{n(1-n)}{2x} + \frac{n(2-3n-2n^2+3n^3)}{24x^2} \right. \\ & \left. + \frac{n^2(-2+n+3n^2-n^3-n^4)}{48x^3} + \dots \right]. \end{aligned} \quad (2.30)$$

Therefore, by matching asymptotic expansions, we have the following.

Corollary 3. (a) $p_3 = \frac{1}{12} = \frac{B_2}{2}$ and $\frac{B_4}{4} = 6p_5 + p_3 - 6p_4$. (b) $\frac{B_{2n}}{2n}$ may be expressed as a sum of p_n values with rational coefficients.

Elaborating part (b), we have the following. We let $S(n, k)$ denote the Stirling numbers of the second kind.

Proposition 1.

$$\begin{aligned} \frac{B_n}{n} &= \sum_{k=1}^n (-1)^k (k-1)! S(n, k) \sum_{\ell=0}^{k-1} p_{\ell+2} \\ &= \sum_{\ell=0}^{n-1} p_{\ell+2} \sum_{k=\ell+1}^n (-1)^k (k-1)! S(n, k). \end{aligned} \quad (2.31)$$

Proof. We let $B_k^{(k)} = (-1)^k \int_0^1 (x)_k dx$ be the Nörlund numbers (e.g., [9, 5]), such that $B_0^{(0)} = 1$, $B_1^{(1)} = -1/2$, and $B_2^{(2)} = 5/6$, and have

$$B_n^{(n)} + nB_{n-1}^{(n-1)} = (-1)^{n+1} n! p_{n+1}. \quad (2.32)$$

By iterating, using the initial value $B_0^{(0)} = 1$, we obtain

$$B_n^{(n)} = (-1)^n n! \left(1 - \sum_{k=0}^{n-1} p_{k+2} \right). \quad (2.33)$$

Substituting into

$$\sum_{k=1}^n S(n, k) \frac{B_k^{(k)}}{k} = -\frac{B_n}{n}, \quad (2.34)$$

and using the sum

$$\sum_{k=1}^n (-1)^k (k-1)! S(n, k) = 0 \quad (2.35)$$

gives the Proposition. \square

The integral representation (2.23) also leads to the following series representation.

Proposition 2. For $\text{Re } y > 0$,

$$\ln \Gamma(y) - y \ln y + y = -\frac{1}{6y} - \frac{1}{2}\psi(y) - \sum_{n=1}^{\infty} [(n+2)p_{n+3} - np_{n+2}] B(y, n+1) + \frac{1}{2} \ln(2\pi). \quad (2.36)$$

Proof. We first integrate (2.23) from $x = 1$ to y , giving

$$\begin{aligned} \ln \Gamma(y) - y \ln y + y - 1 &= - \int_0^1 \frac{[(1-z)^{y-1} - 1]}{\ln(1-z)} \left(\frac{1}{\ln(1-z)} - \frac{1}{z} \right) dz \\ &= - \int_0^1 [(1-z)^{y-1} - 1] \left\{ \sum_{n=1}^{\infty} [(n+1)p_{n+3} - np_{n+2}] z^n + p_3 - \frac{1}{z} + \sum_{n=1}^{\infty} p_{n+1} z^{n-2} \right\} dz \\ &= - \int_0^1 [(1-z)^{y-1} - 1] \left\{ \sum_{n=1}^{\infty} [(n+1)p_{n+3} - np_{n+2}] z^n + p_3 - \frac{1}{2z} + \sum_{n=1}^{\infty} p_{n+2} z^{n-1} \right\} dz, \end{aligned} \quad (2.37)$$

using (1.3) and (1.4). Performing the integration, we find

$$\begin{aligned} \ln \Gamma(y) - y \ln y + y - 1 &= - \sum_{n=1}^{\infty} [(n+1)p_{n+3} - np_{n+2}] \left[B(y, n+1) - \frac{1}{n+1} \right] - p_3 \left(\frac{1}{y} - 1 \right) \\ &\quad - \frac{1}{2} [\psi(y) + \gamma] - \sum_{n=1}^{\infty} p_{n+2} \left[B(y, n) - \frac{1}{n} \right]. \end{aligned} \quad (2.38)$$

Next, the integral representation (1.2) is used for the sums absent the Beta function,

$$\sum_{n=1}^{\infty} \frac{p_{n+2}}{n} = \frac{1}{2} \ln(2\pi) - 1 - \gamma, \quad (2.39)$$

$$\sum_{n=1}^{\infty} p_{n+3} = \frac{5}{12}, \quad - \sum_{n=1}^{\infty} \frac{n}{n+1} p_{n+2} = \gamma - 1. \quad (2.39b)$$

Alternatively, these sums may be obtained by integrating and other otherwise manipulating the generating function (1.3) and taking $z \rightarrow 1$. Then combining terms of (2.38) and (2.39) gives the Proposition. \square

The coefficients p_{n+1} may be readily related to other quantities, including the Bernoulli numbers of the second kind b_n ($n \geq 0$) [8, 10],

$$b_n = \int_0^1 \frac{\Gamma(t+1)}{\Gamma(t-n+1)} dt, \quad (2.40)$$

with $b_0 = 1$, $b_1 = 1/2$, $b_2 = -1/6$, and $b_3 = 1/4$. We have

$$p_{n+1} = -\frac{1}{n!} \int_0^1 \frac{\Gamma(n-t)}{\Gamma(-t)} dt = \frac{(-1)^{n+1}}{n!} \int_0^1 \frac{\Gamma(t+1)}{\Gamma(t+1-n)} dt, \quad (2.41)$$

and hence $p_{n+1} = (-1)^{n-1} b_n / n!$.

The following gives a series representation for $\ln A$. The method of proof shows that a family of such series may be written.

Proposition 3.

$$\ln A = \frac{1}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1}{2} (n+2) p_{n+4} + (n+1) \left(\frac{1}{n} - 1 \right) p_{n+3} + \left[\frac{n^2}{2(n+1)} - 1 + \frac{(6n+1)}{12(n+1)} \right] p_{n+2} \right\}. \quad (2.42)$$

Proof. Let $\zeta(s, a)$ be the Hurwitz zeta function. Then for $\text{Re } s > -(2n - 1)$, $n \in \mathbb{N}_0$, and $\text{Re } a > 0$, there is the integral representation

$$\zeta(s, a) = a^{-s} + \sum_{k=0}^n (s)_{k-1} \frac{B_k}{k!} a^{-k-s+1} + \frac{1}{\Gamma(s)} \int_0^\infty \left(\frac{1}{e^t - 1} - \sum_{k=0}^n \frac{B_k}{k!} t^{k-1} \right) e^{-at} t^{s-1} dt. \quad (2.43)$$

By taking $n = 2$ and $a = 1$, one may find

$$\ln A = \frac{1}{4} + \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) \frac{e^{-t}}{t^2} dt. \quad (2.44)$$

Changing variable with $t = -\ln(1 - z)$ we have

$$\ln A = \frac{1}{4} + \int_0^1 \left(\frac{1}{z} + \frac{1}{\ln(1 - z)} - \frac{1}{2} + \frac{\ln(1 - z)}{12} \right) \frac{dz}{\ln^2(1 - z)}. \quad (2.45)$$

Next the generating functions (1.3), (1.4), and (2.1) are employed in the integrand, and all terms $O(z^{-k})$, $k = 1, 2, 3$ are nullified, as they should. Performing the integration gives

$$\begin{aligned} \ln A = & \frac{1}{4} + \sum_{n=1}^{\infty} \left[\left(\frac{n+1}{n} \right) p_{n+3} - p_{n+2} \right] \\ & + \frac{1}{2} \sum_{n=1}^{\infty} \left[(n+2)p_{n+4} - (2n+1)p_{n+3} + \frac{n^2}{n+1} p_{n+2} \right] \\ & - \frac{1}{2} \sum_{n=1}^{\infty} \left[p_{n+3} - \frac{n}{n+1} p_{n+2} + p_3 \right] + \frac{1}{12} \sum_{n=1}^{\infty} \frac{p_{n+1}}{n}. \end{aligned} \quad (2.46)$$

Shifting the index on the last sum and combining terms gives the Proposition. \square

Noting that $\ln A \simeq 0.248754477033784262547253$, the summation in (2.29) provides an appropriate correction to $1/4$.

Corollary 4. The constant

$$\zeta'(2) = \zeta(2)[\gamma + \ln(2\pi) - 12 \ln A] \quad (2.47)$$

may be written in terms of the coefficients p_n .

Proof. The relation of $\zeta'(2)$ to $\zeta'(-1)$ follows from the functional equation of the Riemann zeta function. The value $\zeta(2)$ may be found in terms of p_n 's via the integral representations (2.34) at $a = 1$. The constant $\ln(2\pi)$ may be written in terms of p_n 's via (1.7) and (1.9). Finally, the Euler constant $\gamma = \sum_{n=1}^{\infty} \frac{p_{n+1}}{n}$, as $\gamma = \int_0^1 \left(\frac{1}{\ln x} + \frac{1}{1-x} \right) dx$.

We may note that more generally we may similarly write series representations for the logarithm of the double Gamma function Γ_2 , since we have for $\text{Re } a > 0$

$$\begin{aligned} \ln \Gamma_2(a) &= \ln A - \frac{a^2}{4} + \left(\frac{a^2}{2} - \frac{a}{2} + \frac{1}{12} \right) \ln a + (1-a) \ln \Gamma(a) \\ &\quad - \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) \frac{e^{-at}}{t^2} dt \\ &= \ln A - \frac{a^2}{4} + \left(\frac{a^2}{2} - \frac{a}{2} + \frac{1}{12} \right) \ln a + (1-a) \ln \Gamma(a) \\ &\quad - \int_0^1 \left(\frac{1}{z} + \frac{1}{\ln(1-z)} - \frac{1}{2} + \frac{\ln(1-z)}{12} \right) (1-z)^{a-1} \frac{dz}{\ln^2(1-z)}. \end{aligned} \quad (2.48)$$

We arrive at

Proposition 4. For $\text{Re } a > 0$,

$$\begin{aligned} \ln \Gamma_2(a) &= \ln A - \frac{a^2}{4} + \left(\frac{a^2}{2} - \frac{a}{2} + \frac{1}{12} \right) \ln a + (1-a) \ln \Gamma(a) \\ &= - \sum_{n=1}^{\infty} \left[(n+2)^2 p_{n+4} - (n+1)(n+2) p_{n+3} + \frac{1}{2} \left(n^2 + n + \frac{1}{6} \right) p_{n+2} \right] B(n+1, a). \end{aligned} \quad (2.49)$$

The details of this calculation are omitted.

We conjecture that the following inequalities hold for the coefficients p_n .

Conjecture 1. (i) For $n \geq 2$,

$$(n+1)p_{n+3} - np_{n+2} < 0, \quad (2.50)$$

(ii) that the sequence $\{p_n\}$ is strictly convex, i.e., for $n \geq 3$,

$$p_n < \frac{1}{2}(p_{n+1} + p_{n-1}), \quad (2.51)$$

(iii) that the sequence $\{p_n\}$ is strictly log-convex, i.e., for $n \geq 3$,

$$p_n^2 < p_{n-1}p_{n+1}. \quad (2.52)$$

The convexity and log-convexity themselves are probably not difficult to show, and (iii) implies (ii). Certainly these inequalities hold for n sufficiently large. Using the known asymptotic form $p_n \sim 1/[n(\ln n + \gamma)^2]$, the leading asymptotic form of the difference of the left and right sides of the Conjecture is given by: for (i), $-2/[n(\ln n + \gamma)^3]$, for (ii), $-\ln^2 n/[n^3(\ln n + \gamma)^4]$, and for (iii), $-\ln^2 n/[n^4(\ln n + \gamma)^6]$.

Expressions for the Stieltjes constants and $\zeta(m, a)$ values

The Hurwitz zeta function, defined by $\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$ for $\text{Re } s > 1$ and $\text{Re } a > 0$ extends to a meromorphic function in the entire complex s -plane. In the Laurent expansion

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(a) (s-1)^n, \quad (3.1)$$

$\gamma_n(a)$ are the Stieltjes constants (e.g., [3, 4]), and by convention one takes $\gamma_k = \gamma_k(1)$. One has $\gamma_0(a) = -\psi(a)$ and $\gamma_0 = \gamma$. The Stieltjes constants may be expressed via

the limit formula

$$\gamma_k(a) = \lim_{N \rightarrow \infty} \left[\sum_{n=0}^N \frac{\ln^k(n+a)}{n+a} - \frac{\ln^{k+1}(N+a)}{k+1} \right].$$

We dispense with further preliminaries concerning the $\gamma_n(a)$'s.

We may write series and integral representations for these constants based upon the p_n coefficients. This development is illustrated in the next result.

Proposition 5. (a)

$$\begin{aligned} -\frac{1}{2} \ln^2 a - \gamma_1(a) &= \gamma[\ln a - \psi(a)] + \sum_{n=1}^{\infty} p_{n+1} \int_0^1 u^{a-1} \ln(-\ln u) (1-u)^{n-1} du \\ &= \gamma[\ln a - \psi(a)] + \int_0^1 u^{a-1} \ln(-\ln u) \left[\frac{1}{1-u} + \frac{1}{\ln u} \right] du, \end{aligned} \quad (3.2a)$$

and

$$\begin{aligned} -\gamma_1 - \gamma^2 &= \sum_{n=1}^{\infty} p_{n+1} \int_0^{\infty} e^{-t} \ln t (1-e^{-t})^{n-1} dt \\ &= - \int_0^{\infty} \left[\frac{1}{1-e^t} + \frac{e^{-t}}{t} \right] \ln t \, dt, \end{aligned} \quad (3.2b)$$

(b)

$$\gamma_2 = -\gamma(\gamma^2 + \zeta(2) + 2\gamma_1) + \int_0^{\infty} (\ln^2 t) e^{-t} \sum_{n=1}^{\infty} p_{n+1} (1-e^{-t})^{n-1} dt, \quad (3.3)$$

and (c)

$$\begin{aligned} -\gamma_3 &= \gamma^4 + \frac{\pi^2}{2} \gamma_1 + \frac{\gamma^2}{2} (\pi^2 + 6\gamma_1) + 3\gamma\gamma_2 + 2\gamma\zeta(3) \\ &\quad + \int_0^{\infty} (\ln^3 t) e^{-t} \sum_{n=1}^{\infty} p_{n+1} (1-e^{-t})^{n-1} dt. \end{aligned} \quad (3.4)$$

Proof. For $\operatorname{Re} s > 0$ and $\operatorname{Re} a > 0$ we have the integral representation

$$\zeta(s, a) - \frac{1}{(s-1)} \frac{1}{a^{s-1}} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-ax} x^{s-1} \left(\frac{1}{1-e^{-x}} - \frac{1}{x} \right) dx$$

$$\begin{aligned}
&= \frac{1}{\Gamma(s)} \int_0^1 (1-z)^{a-1} [-\ln(1-z)]^{s-1} \left[\frac{1}{z} + \frac{1}{\ln(1-z)} \right] dz \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} p_{n+1} \int_0^1 u^{a-1} (-\ln u)^{s-1} (1-u)^{n-1} du.
\end{aligned} \tag{3.5}$$

It is readily found that the limit of (3.5) as $s \rightarrow 1$ agrees with Lemma 2, and this result is used in all parts of the Proposition. For parts (a)-(c) we take successive derivatives of (3.5) with respect to s and evaluate at $s = 1$. For the integral representations in (a) we use the integral representation for p_{n+1} in (1.2). \square

The following further emphasizes the connection of the p_{n+1} coefficients with analytic number theory. We let ψ' be the trigamma function, $\psi^{(j)}$ the polygamma functions, and $H_n = \sum_{k=1}^n \frac{1}{k} = \psi(n+1) + \gamma$ be the n th harmonic number.

Corollary 5.

$$\zeta(2) = 1 + \sum_{n=1}^{\infty} \frac{p_{n+1}}{n} H_n, \tag{3.6}$$

and

$$\zeta(3) = \frac{1}{2} + \frac{\zeta(2)}{2} \gamma + \frac{1}{2} \sum_{n=1}^{\infty} \frac{p_{n+1}}{n} [H_n^2 - \psi'(n+1)]. \tag{3.7}$$

Proof. This follows from the special case of (3.5),

$$\zeta(s) - \frac{1}{(s-1)} = \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} p_{n+1} \int_0^1 (-\ln u)^{s-1} (1-u)^{n-1} du. \tag{3.8}$$

The integrals are given by

$$\int_0^1 (-\ln u)^{s-1} (1-u)^{n-1} du = (-1)^{s-1} \partial_x^{s-1} B(x, n)|_{x=1}. \tag{3.9}$$

In writing (3.7) we have used the previously given summation expression for γ . \square

Similarly further values of $\zeta(n)$ may be written. We may note the appearance of generalized harmonic numbers $H_n^{(r)} = \sum_{k=1}^n \frac{1}{k^r}$, $H_n \equiv H_n^{(1)}$. In particular, as regards (3.7), $H_n^{(2)} = -[\psi'(n+1) - \psi'(1)] = -[\psi'(n+1) - \zeta(2)]$, so that

$$\zeta(3) = \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{p_{n+1}}{n} [H_n^2 + H_n^{(2)}]. \quad (3.10)$$

Furthermore,

$$\zeta(4) = \frac{1}{3} + \frac{1}{6} \sum_{n=1}^{\infty} \frac{p_{n+1}}{n} [H_n^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}]. \quad (3.11)$$

As in Corollary 5, the summand is positive and the convergence is from below.

The generalized harmonic numbers are given by

$$H_n^{(r)} = \frac{(-1)^{r-1}}{(r-1)!} [\psi^{(r-1)}(n+1) - \psi^{(r-1)}(1)]. \quad (3.12)$$

We may systematize Corollary 5 and related results in the following manner. We introduce the (exponential) complete Bell polynomials $Y_n = Y_n(x_1, x_2, \dots, x_n)$ appearing in the expansion

$$\exp \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right) = 1 + \sum_{n=1}^{\infty} Y_n(x_1, x_2, \dots, x_n) \frac{t^n}{n!}, \quad Y_0 = 1. \quad (3.13)$$

Then

Proposition 6. For $m \in \mathbb{N}$, $m \geq 2$, (a)

$$\zeta(m) = \frac{1}{m-1} + \frac{1}{(m-1)!} \sum_{n=1}^{\infty} \frac{p_{n+1}}{n} Y_{m-1}(H_n, H_n^{(2)}, 2!H_n^{(3)}, \dots, (m-2)!H_n^{(m-1)}), \quad (3.14)$$

and (b) for $\text{Re } a > 0$,

$$\zeta(m, a) = \frac{1}{(m-1)} \frac{1}{a^{m-1}} + \frac{(-1)^{m-1}}{(m-1)!} \sum_{n=1}^{\infty} p_{n+1} B(a, n) Y_{m-1}[\psi(a) - \psi(a+n), \psi'(a) - \psi'(a+n), \dots,$$

$$\psi^{(m-1)}(a) - \psi^{(m-1)}(a+n)]. \quad (3.15)$$

Proof. We apply (e.g., [6])

Lemma 4. For differentiable functions f and g such that $f'(x) = f(x)g(x)$, assuming all higher order derivatives exist, we have

$$\left(\frac{d}{dx}\right)^j f(x) = f(x)Y_j[g(x), g'(x), \dots, g^{(j-1)}(x)]. \quad (3.16)$$

We put $f(x) = B(x, n)$ and $g(x) = \frac{d}{dx} \ln B(x, n) = \psi(x) - \psi(x+n)$. Then by (3.12) $g^{(r)}(1) = (-1)^r(r-1)!H_n^{(r)}$. Part (a) then follows from (3.8) and (3.9) with $s = m$ and $B(1, n) = 1/n$.

Similarly for part (b) we use (3.5), in which the integral on the right side is given by $(-1)^{s-1}\partial_a^{s-1}B(a, n)$. \square

Alternatively, the integral on the right side of (3.5) may be treated with a generating function for the Stirling numbers of the first kind [1] (p. 824) so that

$$\int_0^1 (1-u)^{a-1} [-\ln(1-u)]^{m-1} u^{n-1} du = (-1)^{m-1} (m-1)! \sum_{j=m-1}^{\infty} \frac{(-1)^j}{j!} s(j, m-1) B(n+j, a). \quad (3.17)$$

Additional sums

We collect the following summations with the p_{n+1} coefficients.

Proposition 7.

$$\sum_{n=1}^{\infty} \frac{p_{n+1}}{n+a} = \frac{1}{a} - \int_0^1 B(a, x+1) dx, \quad (3.18)$$

in particular

$$\sum_{n=1}^{\infty} \frac{p_{n+1}}{n+1} = 1 - \ln 2, \quad (3.19)$$

$$\sum_{n=1}^{\infty} \frac{p_{n+1}}{n^2} = \frac{1}{2}(\gamma^2 - 1) + \frac{\pi^2}{12} + \frac{1}{2} \int_0^1 \psi^2(x+1) dx, \quad (3.20)$$

$$\sum_{n=1}^{\infty} \frac{p_{n+1}}{n^3} = \frac{1}{12}[-5 + 2\gamma^3 + \gamma\pi^2 + 4\zeta(3)] + \frac{1}{6} \int_0^1 [3\gamma\psi^2(x+1) + \psi^3(x+1)] dx, \quad (3.21)$$

and for $j \in \mathbb{N}^+$,

$$\sum_{n=1}^{\infty} \frac{p_{n+1}}{n^j} z^n = z \int_0^1 x {}_{j+2}F_{j+1}(1, 1, \dots, 1, 1-x; 2, 2, \dots, 2; z) dx. \quad (3.22)$$

Proof. These may be obtained with the aid of the integral representation of (1.2), and we omit further details. \square

As concerns the integral on the right side of (3.20), we have the following result.

Corollary 6.

$$\int_0^1 \psi^2(x+1) dx = 2\gamma_1 - \frac{\pi^2}{3} + \int_0^1 \left[2\frac{\psi(x)}{x} + \frac{1}{x^2} - 2\gamma\psi(x) + \psi'(x) \right] dx. \quad (3.23)$$

Proof. We use Proposition 3(b) of [5],

$$\gamma_1 = \frac{\pi^2}{6} + \int_0^1 \left(\gamma\psi(x) + \frac{1}{2}[\psi^2(x) - \psi'(x)] \right) dx, \quad (3.24)$$

together with $\psi^2(x+1) = \psi^2(x) + 2\frac{\psi(x)}{x} + \frac{1}{x^2}$. \square

Based upon various representations of the Beta function, we may collect the following representations for the integral occurring in (3.5) for the Riemann zeta function case, $a = 1$. It should be clear which of these are restricted to $s \geq 1$ an integer.

Proposition 8.

$$I_n(s) \equiv \int_0^1 (-\ln u)^{s-1} (1-u)^{n-1} du = (-1)^{s-1} \partial_a^{s-1} B(a, n) \Big|_{a=1}$$

$$\begin{aligned}
&= \frac{\Gamma(s)}{n} \sum_{k=1}^n \frac{(-1)^{k-1}}{k^{s-1}} \binom{n}{k} \\
&= \frac{\Gamma(s)}{n} \sum_{1 \leq j_1 \leq \dots \leq j_{s-1} \leq n} \frac{1}{j_1 \cdots j_{s-1}} \\
&= \Gamma(s) \int_{[0,1]^s} (1 - x_1 x_2 \cdots x_s)^{n-1} dx_1 \cdots dx_s \\
&= \frac{1}{2\pi i} \frac{\Gamma(s)}{n} \oint_{|z|=r < 1} \frac{1}{z^s} \prod_{j=1}^n \frac{dz}{(1 - z/j)} \\
&= \frac{(-1)^n}{2\pi i} \frac{\Gamma(s)}{n} \int_{1/2-i\infty}^{1/2+i\infty} \frac{n!}{y^s (y-1) \cdots (y-n)} dy. \tag{3.25}
\end{aligned}$$

Furthermore, for $s \geq 2$,

$$\frac{(s-1)}{n} I_n(s) = \left(\frac{s-1}{n-1} \right) I_{n-1}(s) + \frac{1}{n} I_n(s-1), \tag{3.26a}$$

and

$$\frac{(s-1)}{n} I_n(s) = \sum_{j=1}^n \frac{I_j(s-1)}{j^2}. \tag{3.26b}$$

Upper bound for p_{n+1} and other results, including polylogarithmic representation of the Stieltjes constants

We let $\text{Si}(z) = \int_0^z \frac{\sin t}{t} dt$ be the sine integral.

Proposition 9. For $n \geq 1$,

$$p_{n+1} < -\frac{1}{2} + \frac{1}{\pi} \text{Si}(\pi) + (-1)^{n-1} \left[\frac{1}{2} - \frac{\text{Si}[(n-1)\pi]}{\pi} \right]. \tag{4.1}$$

Proof. Based upon contour integration, Knessl has shown the following integral representation [3],

$$p_{n+1} = \int_0^\infty \frac{1}{(1+u)^n} \frac{du}{(\ln^2 u + \pi^2)}, \quad n \geq 1. \tag{4.2}$$

Then

$$\begin{aligned}
p_{n+1} &= \int_0^1 \frac{1}{(1+u)^n} \frac{du}{(\ln^2 u + \pi^2)} + \int_1^\infty \frac{1}{(1+u)^n} \frac{du}{(\ln^2 u + \pi^2)} \\
&< \int_0^1 \frac{du}{(\ln^2 u + \pi^2)} + \int_1^\infty \frac{du}{u^n (\ln^2 u + \pi^2)} \\
&= \int_1^\infty \left(\frac{1}{u^2} + \frac{1}{u^n} \right) \frac{du}{(\ln^2 u + \pi^2)} \\
&= \int_0^\infty \frac{[1 + e^{-(n-1)v}]}{v^2 + \pi^2} dv \\
&= -\frac{1}{2} + \frac{1}{\pi} \text{Si}(\pi) + (-1)^{n-1} \left[\frac{1}{2} - \frac{\text{Si}[(n-1)\pi]}{\pi} \right].
\end{aligned} \tag{4.3}$$

The integral of (4.3) could be evaluated by means of contour integration, but we supply another means. We put

$$I(k) \equiv \int_0^\infty \frac{e^{-(k-1)v}}{v^2 + \pi^2} dv, \tag{4.4}$$

and form

$$I''(k) + \pi^2 I(k) = \int_0^\infty e^{-(k-1)v} dv = \frac{1}{k-1}. \tag{4.5}$$

The homogeneous solutions of this differential equation are of course $\cos k\pi$ and $\sin k\pi$ and their constant Wronskian is $W = \pi$. Per variation of parameters, a particular solution then takes the form

$$I_p(k) = -\frac{\cos \pi k}{\pi} \int \frac{\sin \pi k}{k-1} dk + \frac{\sin \pi k}{\pi} \int \frac{\cos \pi k}{k-1} dk.$$

Then solving (4.5) subject to $I(1) = 1/2$ gives

$$I(k) = \frac{1}{\pi} \{ -\sin(k\pi) \text{Ci}[(k-1)\pi] + \cos(k\pi) \text{Si}[(k-1)\pi] \} - \frac{1}{2} \cos k\pi + c_2 \sin k\pi, \tag{4.6}$$

where $\text{Ci}(z) = -\int_z^\infty \frac{\cos t}{t} dt$ is the cosine integral. Then imposing $I(\infty) = 0$ gives the constant $c_2 = 0$. \square

Asymptotically, as $n \rightarrow \infty$ on the right side of (4.1), this upper bound is $\frac{\text{Si}(\pi)}{\pi} - \frac{1}{2} + \frac{1}{\pi^2 n} + O\left(\frac{1}{n^2}\right)$.

We may extend Lemma 3 to the following.

Lemma 5.

$$\begin{aligned}
\psi(x) - \ln x &= -\sum_{n=1}^{\infty} p_{n+1} \frac{(n-1)!}{(x)_n} = -\frac{1}{2x} - 2 \int_0^\infty \frac{t dt}{(t^2 + x^2)(e^{2\pi t} - 1)} \\
&= -\frac{1}{2x} - 2 \int_0^\infty \frac{v dv}{(1 + v^2)(e^{2\pi x v} - 1)} \\
&= -\frac{1}{2x} - 2 \int_1^\infty \frac{\ln u \, du}{u(1 + \ln^2 u)(u^{2\pi x} - 1)} \\
&= -\frac{1}{x} \int_0^\infty \frac{{}_2F_1(1, 1; x+1; v) dv}{v [\ln^2(\frac{1}{v} - 1) + \pi^2]}. \tag{4.7}
\end{aligned}$$

Proof. The first line follows from a known integral representation of $\psi(x) - \ln x$ [7] (p. 943). The final equality follows from the representation (4.2) for p_{n+1} . \square

For the next result we introduce the polylogarithm function $\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$ that may be analytically continued to the whole complex plane. This function has the integral representation for $\text{Re } s > 0$

$$\text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} dt}{e^t - z}, \tag{4.8}$$

and we note that $\text{Li}_1(z) = -\ln(1 - z)$. The branch cut for $\text{Li}_s(z)$ in the complex z plane may be taken from 1 to ∞ . This function is also given when s is a positive integer by $\text{Li}_m(z) = z {}_{m+1}F_m(1, 1, \dots, 1; 2, \dots, 2; z)$, and this provides another way of seeing the branch point at $z = 1$.

Proposition 10.

$$\gamma_1 + \gamma^2 = \int_0^\infty \left[\gamma \ln \left(1 + \frac{1}{u} \right) + \partial_s|_{s=1} \text{Li}_s \left(-\frac{1}{u} \right) \right] \frac{du}{(\ln^2 u + \pi^2)}, \quad (4.9)$$

and

$$\begin{aligned} & \gamma_2 + \gamma[\gamma^2 + \zeta(2) + 2\gamma_1] \\ = & \int_0^\infty \left[(\gamma^2 + \zeta(2)) \ln \left(1 + \frac{1}{u} \right) + 2\gamma \partial_s|_{s=1} \text{Li}_s \left(-\frac{1}{u} \right) - \partial_s^2|_{s=1} \text{Li}_s \left(-\frac{1}{u} \right) \right] \frac{du}{(\ln^2 u + \pi^2)}. \end{aligned} \quad (4.10)$$

Proof. By (3.2b) and the representation (4.2),

$$\begin{aligned} -\gamma_1 - \gamma^2 &= \sum_{n=1}^\infty \int_0^\infty \frac{1}{(1+u)^n} \frac{du}{(\ln^2 u + \pi^2)} \int_0^\infty e^{-t} \ln t (1 - e^{-t})^{n-1} dt \\ &= \int_0^\infty \int_0^\infty \frac{du}{(1+e^t u)} \ln t \, dt. \end{aligned}$$

The interchanges are justified by the absolute convergence of the integrals. Then (4.9) follows from logarithmic differentiation of the integral of (4.8). Similarly for (4.10), (3.3) is used along with (4.2). \square

Several versions of (4.10) may be written by employing (4.9).

Proposition 10 is clarified and generalized with the following.

Proposition 11. We have

$$\zeta(s) - \frac{1}{s-1} = - \int_0^\infty \frac{\text{Li}_s(-v) dv}{v^2 (\ln^2 v + \pi^2)}. \quad (4.11)$$

Consequently, for $k \geq 0$,

$$\gamma_k = (-1)^{k-1} \left(\frac{\partial}{\partial s} \right)^k \bigg|_{s=1} \int_0^\infty \frac{\text{Li}_s(-v) dv}{v^2 (\ln^2 v + \pi^2)}. \quad (4.12)$$

Proof. We first combine the representation (3.5) with $a = 1$ with the second line of Proposition 8 resulting from binomial expansion:

$$\begin{aligned}
\zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} \frac{p_{n+1}}{n} \sum_{k=1}^n \binom{n}{k} \frac{(-1)^{k-1}}{k^{s-1}} \\
&= \sum_{n=0}^{\infty} \frac{p_{n+2}}{(n+1)} \sum_{k=0}^n \binom{n+1}{k+1} \frac{(-1)^k}{(k+1)^{s-1}} = \sum_{n=0}^{\infty} p_{n+2} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k}{(k+1)^s} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} p_{n+2} \binom{n}{k} \frac{(-1)^k}{(k+1)^s}.
\end{aligned} \tag{4.13}$$

We now use the representation (4.2) for p_{n+2} , so that

$$\begin{aligned}
\zeta(s) - \frac{1}{s-1} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{u^{k+1}} \frac{1}{(k+1)^s} \frac{du}{(\ln^2 u + \pi^2)} = - \int_0^{\infty} \frac{\text{Li}_s(-\frac{1}{u})}{\ln^2 u + \pi^2} du \\
&= - \int_0^{\infty} \frac{\text{Li}_s(-v) dv}{v^2(\ln^2 v + \pi^2)}.
\end{aligned} \tag{4.14}$$

(4.12) then immediately follows. \square

Example. When $k = 0$ in (4.11) and $s \rightarrow 1$,

$$\gamma_0 = \gamma = \int_0^{\infty} \frac{\ln(1+v)}{v^2(\ln^2 v + \pi^2)} dv, \tag{4.15}$$

and this case is equivalent to (2.87) in [3].

Let Φ denote the Lerch zeta function, $\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$, and analytically continued. This series holds for $s \in \mathbb{C}$ when $|z| < 1$ and for $\text{Re } s > 1$ when $|z| = 1$. The function Φ has an integral representation

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(a-1)t}}{e^t - z} dt,$$

for $\operatorname{Re} s > 0$ when $|z| \leq 1$, $z \neq 1$, and for $\operatorname{Re} s > 1$ when $z = 1$.

Proposition 12.

$$\zeta(s, a) - \frac{1}{(s-1)} \frac{1}{a^{s-1}} = \int_0^\infty \frac{\Phi(-v, s, a)}{v(\ln^2 v + \pi^2)} dv. \quad (4.16)$$

Proof sketch. Now

$$\int_0^1 u^{a-1} (-\ln u)^{s-1} (1-u)^{n-1} du = \frac{\Gamma(s)}{n} \sum_{k=1}^n \frac{(-1)^{k-1} k}{(k+a-1)^s} \binom{n}{k}. \quad (4.17)$$

Then (3.5) is used, following steps similar to the proof of Proposition 11, so that

$$\zeta(s, a) - \frac{1}{(s-1)} \frac{1}{a^{s-1}} = \sum_{k=0}^\infty \sum_{n=k}^\infty p_{n+2} \binom{n}{k} \frac{(-1)^k}{(k+a)^s}. \quad (4.18)$$

Performing the sum on k and changing variable in the integral gives (4.16). \square

Corollary 7.

$$\ln a - \psi(a) = \ln a + \gamma_0(a) = \int_0^\infty \frac{\Phi(-v, 1, a)}{v(\ln^2 v + \pi^2)} dv. \quad (4.19)$$

Appendix

Here we show that for $n \in \mathbb{N}^+$

$$\frac{(a)_j}{(a+n)_j} = \frac{(a)_n}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{(a+k)} \frac{(a+k)_j}{(a+k+1)_j}. \quad (A.1)$$

Proof. We have the known decomposition

$$\frac{1}{x(x+1) \cdots (x+N)} = \frac{1}{N!} \sum_{k=0}^N \binom{N}{k} \frac{(-1)^k}{(x+k)}. \quad (A.2)$$

Then

$$\frac{(a)_j}{(a+n)_j} = \frac{a(a+1)(a+2) \cdots (a+n-1)}{(a+j)(a+j+1) \cdots (a+j+n-1)}$$

$$= \frac{(a)_n}{(n-1)!} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{(a+j+k)}. \quad (A.3)$$

By noting that

$$\frac{(a+k)_j}{(a+k+1)_j} = \frac{(a+k)}{(a+k+j)}, \quad (A.4)$$

the result follows. \square

References

- [1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Washington, National Bureau of Standards (1964).
- [2] T. Amderberhan et al., Integrals of powers of loggamma, Proc. AMS **139**, 535-545 (2011).
- [3] M. W. Coffey, Series representations for the Stieltjes constants, arXiv:0905.1111 (2009), to appear in Rocky Mtn. J. Math.
- [4] M. W. Coffey, Addison-type series representation for the Stieltjes constants, J. Num. Th. **130**, 2049-2064 (2010), arXiv:0912.2391.
- [5] M. W. Coffey, Series representations of the Riemann and Hurwitz zeta functions and series and integral representations of the first Stieltjes constant, arXiv:1106.5147 (2011).
- [6] M. W. Coffey, A set of identities for a class of alternating binomial sums arising in computing applications, Util. Math. **76**, 79 (2008).
- [7] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York (1980).
- [8] C. Jordan, Calculus of finite differences, Chelsea (1979).
- [9] N. E. Nörlund, Vorlesungen Über Differenzenrechnung, Springer (1924).

- [10] S. Roman, The umbral calculus, Dover (2005).